Static Games of Incomplete Information

Introduction

In many economically important situations the game may begin with some player having private information about something relevant to her decision making. These are called games of *incomplete information*, or *Bayesian* games. (Incomplete information is not to be confused with *imperfect* information in which players do not perfectly observe the actions of other players.) For example in a currently monopolized market the incumbent firm knows its marginal cost, which would influence its production decision if a potential rival entered the market, but the rival does not know the incumbent's cost. Or in a sealed-bid auction each player knows how much she values the object for sale but does not know her opponents' valuations. Although any given player does not know the private information of an opponent, she will have some beliefs about what the opponent knows, and we will assume that these beliefs are common knowledge.

In many cases of interest we will be able to model the informational asymmetry by specifying that each player knows her own payoff function, but that she is uncertain about what her opponents' payoff functions are. In the monopolized market example above the incumbent's knowledge of its own costs translates into knowledge of its profit for any given combination of production decisions by the two firms. In the sealed-bid auction example, knowing her own valuation of the object is equivalent to knowing her utility if she is the successful bidder for any given price paid, and therefore to knowing her utility for any set of players' bids.

We will introduce the notion of a player's *type* to describe her private information. A player's type fully describes any information she has which is not common knowledge. A player may have several types—even an infinity of types, one for each possible state of her private information. Each player knows her own type with complete certainty. Her beliefs about other players' types are captured by a common-knowledge joint probability distribution over the others' types. In the currently monopolized-market example, if the incumbent firm's cost were restricted to being either "low" or "high," it would have two types, e.g. $\{\underline{c},\bar{c}\}$. If its cost could be any value in some interval $[\underline{c},\bar{c}]$, it would have a continuum of types.

We can think of the game as beginning with a move by Nature, who to each player assigns a type. Nature's move is imperfectly observed, however: each player observes the type which Nature has bestowed upon her, but no player directly observes the type bestowed upon any other player. We can think of the game which follows as being played by a single type of each player, where at least one player doesn't know which type of some other player she is facing.

A Bayesian game

Let $I=\{1,\ldots,n\}$ be the set of players. We refer to a type of player i by θ_i, where this type is a member of player i's *type space* Θ_i; i.e. $\theta_i\in\Theta_i$. We denote an n-tuple of types, one for each player—or *type profile*—by $\theta=(\theta_1,\ldots,\theta_n)\in\Theta\equiv\times_{i\in I}\Theta_i$, where Θ is the *type-profile space*. When we focus on the types of a player's opponents, we consider deleted type profiles of the form $\theta_{-i}=(\theta_1,\ldots,\theta_{i-1},\theta_{i+1},\ldots,\theta_n)\in\Theta_{-i}$.

For our current purposes we will consider the game played after Nature's type assignments as one in strategic form; we are considering static (i.e. simultaneous-move) games of incomplete information.[1] At a later time we will extend this framework to include dynamic games.

The "bigger" game begins when Nature chooses each player's type and reveals it to her alone. In the strategic-form game which follows, each player i ultimately chooses some pure action $a_i\in A_i$. The n-tuple of actions chosen is the action profile $a\in A\equiv\times_{i\in I}A_i$.[2] The payoff player i receives in general depends on the actions a of all players as well as the types θ of all players; i.e. $u_i(a,\theta)$, where $u_i\colon A\times\Theta\to\mathbb{R}$.[3] For most of our discussion we will assume that the action and type spaces are finite sets. We denote by $\mathscr{A}_i\equiv\Delta(A_i)$ the space of player-i mixed actions. A typical mixed action for player i is $\alpha_i\in\mathscr{A}_i$. A typical deleted mixed-action profile by player i's opponents is $\alpha_{-i}\in\mathscr{A}_{-i}\equiv\times_{j\in I\setminus\{i\}}\mathscr{A}_j$; also $\mathscr{A}\equiv\times_{i\in I}\mathscr{A}_i$.

Beliefs

We assume that there is an objective probability distribution $p\in\Delta(\Theta)$ over the type space Θ, which Nature consults when assigning types.[4] In other words, the probability with which Nature draws the type profile $\theta=(\theta_1,\ldots,\theta_n)$—and hence assigns type θ_1 to player 1, type θ_2 to player 2, etc.—is $p(\theta)$. The

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1 See Fudenberg and Tirole [1991: Chapter 6] and/or Gibbons [1992: Chapter 3] for more on static Bayesian games.

2 When studying repeated games we reserved the symbol s to refer to strategies more complicated than stage-game actions, viz. for repeated-game strategies, which were sequences of history-dependent stage-game actions. We do this again, letting a denote actions in the strategic-form game succeeding Nature's revelation of types, and reserving s to refer to a more complicated strategic object, which depends on type, which we'll need in the larger game.

3 It's easy to see why other players' actions should enter into player i's payoff function, but why should other players' *types* enter into player i's payoffs? Sure, another player's type can influence his action; but this indirect influence of another's type on i's payoff would be captured by the direct effect the other's action has on i's payoff. A reason we should allow in general for a direct dependence of player i's payoff upon others' types is given by the following example. Assume that one firm j has private information about demand, which therefore is captured by his type θ_j. When firm i competes with firm j, the market outcome and hence firm i's profit will depend on the demand and therefore on firm j's type.

4 For any finite set T we denote by $\Delta(T)$ the set of probability distributions over T.

marginal distribution of player i's type is $p_i \in \Delta(\Theta_i)$, where

$$p_i(\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_i, \theta_{-i}). \tag{1}$$

This number $p_i(\theta_i)$ is the probability that Nature draws a type profile θ whose i-th component is θ_i.[5] For later technical convenience we remove from a player's type space any impossible types; i.e. we require, for all $i \in I$ and all $\theta_i \in \Theta_i$, that $p_i(\theta_i) > 0$.

We assume that the probability distribution p, which generates the type profile, is common knowledge and that each player derives from p her subjective beliefs about the types of her opponents. We represent these beliefs by a conditional probability $\hat{p}_i(\theta_{-i} | \theta_i)$ that the opponents' types are a particular deleted type profile $\theta_{-i} \in \Theta_{-i}$ given that player i's known type is θ_i. By Bayes' Rule we have[6]

$$\hat{p}_i(\theta_{-i} | \theta_i) = \frac{\mathbb{P}(\theta_i \,\&\, \theta_{-i})}{\mathbb{P}(\theta_i)} = \frac{p(\theta)}{p_i(\theta_i)}. \tag{2}$$

Player i's knowledge of her own type may or may not affect her beliefs about the types of her opponents. When players' types are independent, the probability of a particular type profile θ is just the product of the players' marginal distributions, each evaluated at the type θ_j specified by θ, i.e. $\forall \theta \in \Theta$,

$$p(\theta) = \prod_{j \in I} p_j(\theta_j). \tag{3}$$

Therefore, when players' types are independent, player i's subjective beliefs about others' types are independent of her own type:

$$\hat{p}_i(\theta_{-i} | \theta_i) = \frac{p(\theta)}{p_i(\theta_i)} = \frac{\prod_{j \in I} p_j(\theta_j)}{p_i(\theta_i)} = \prod_{j \in I \setminus \{i\}} p_j(\theta_j), \tag{4}$$

which does not involve θ_i.

Example: Micro type spaces

Consider two players, Lucky and Dopey. They are in separate rooms and will compete against each other in a computer-intensive task. Clearly the brand of computer each employs will significantly affect her performance. Each player observes the brand of her own computer but not that of her opponent, so we describe each player's type by the brand of the computer with which she is endowed. The two available brands are Macintosh and IBM. Therefore both players have identical type spaces

5 The i subscript on the p_i does not mean that we are referring to player i's beliefs. Indeed, player i is the only player guaranteed to know player i's type. The subscript actually serves only a formal requirement: identifying the domain of this function, viz. Θ_i, and the deleted type profile over which the summation should extend.

6 Here the subscript i on the p_i happens to identify the holder of these beliefs; however, it actually refers to the component of θ on which this probability is conditioned. Note also that the denominator is positive, and therefore this quantity is well defined, because we have excluded impossible types.

$$\Theta_L = \Theta_D = \{\text{Mac}, \text{IBM}\}.$$

The type-profile space is

$$\Theta = \Theta_L \times \Theta_D = \{(\text{Mac}, \text{Mac}), (\text{Mac}, \text{IBM}), (\text{IBM}, \text{Mac}), (\text{IBM}, \text{IBM})\}.$$

First assume computers are being handed out by independent fair-coin tosses such that heads dispenses a Macintosh and tails an IBM. In this scenario the types are independent; e.g. $p((\text{Mac}, \text{IBM})) = p_L(\text{Mac}) p_D(\text{IBM}) = \frac{1}{4}$. Lucky's prior subjective beliefs—prior, that is, to observing her own type—about Dopey's computer are given by $p_D(\text{Mac}) = p((\text{Mac}, \text{Mac})) + p((\text{IBM}, \text{Mac})) = \frac{1}{2}$.[7] After observing that she was awarded a Mac, her posterior beliefs about Dopey's computer are $\hat{p}_L(\text{Mac}|\text{Mac}) = p((\text{Mac}, \text{Mac}))/p_L(\text{Mac}) = p_L(\text{Mac}) p_D(\text{Mac})/p_L(\text{Mac}) = p_D(\text{Mac}) = \frac{1}{2}$; i.e. the prior and the posterior are the same.

Alternatively, assume that there is only one Mac, and it is awarded to Lucky or Dopey on the basis of a coin toss. The unfortunate remaining player receives an IBM. In this case the probability distribution over the type-profile space is

$$p((\text{Mac}, \text{IBM})) = p((\text{IBM}, \text{Mac})) = \tfrac{1}{2}, \quad p((\text{Mac}, \text{Mac})) = p((\text{IBM}, \text{IBM})) = 0.$$

Although Lucky's prior about Dopey's computer is still $p_D(\text{Mac}) = p_D(\text{IBM}) = \frac{1}{2}$, after Lucky observes that she was awarded the Mac, she updates her prior to reflect her certainty that Dopey received the IBM:

$$\hat{p}_L(\text{IBM} \mid \text{Mac}) = \frac{p((\text{Mac}, \text{IBM}))}{p_L(\text{Mac})} = \frac{1/2}{1/2} = 1.$$

This joint probability distribution and the associated marginal probabilities are shown in the table below.

		Dopey: Mac	Dopey: IBM	
Lucky	Mac	0	1/2	$1/2 = p_L(\text{Mac})$
	IBM	1/2	0	$1/2 = p_L(\text{IBM})$
		$1/2 = p_D(\text{Mac})$	$1/2 = p_D(\text{IBM})$	

Strategies

We will now see that it is not sufficient for a strategy for player $i \in I$ in a Bayesian game to merely specify an action for that player; it must specify an action for every type $\theta_i \in \Theta_i$ of player i. Player i's payoff function depends upon her type. Therefore for given actions by, and types of, her opponent, each type of player i will be solving a different maximization problem yielding different best responses; each

7 Note, again, that the D subscript on the p_D refers to the component of θ with respect to which the marginal distribution is computed, not to the player whose beliefs we are discussing.

type of player i is playing a different game from her sisters. Therefore a pure strategy for player i in a static Bayesian game is *type contingent*; it is a function $s_i{:}\,\Theta_i\to A_i$. The space of all such functions and hence player i's pure-strategy space is $S_i=A_i^{\Theta_i}$.[8] For a particular type θ_i of player i, her strategy s_i specifies some action $a_i=s_i(\theta_i)\in A_i$. A mixed strategy $\sigma_i{:}\,\Theta_i\to\mathcal{A}_i$ for player i specifies a mixed action $\alpha_i\in\mathcal{A}_i$ for each type of player i; i.e. $\forall\theta_i\in\Theta_i$, $\sigma_i(\theta_i)\in\mathcal{A}_i$. As usual $\Sigma_i=\mathcal{A}_i^{\Theta_i}$ is the space of all player-i mixed strategies.

At this point you might say: "Fine. I agree that different types would play different actions; but, since only one type of each player is participating in the game at any time, it is only necessary for an equilibrium to specify *that* type's action." In order to rebut this line of reasoning let's consider some player j, who wants to play a best response to i's strategy. The problem for player j is that player i's action will depend upon her type θ_i, and player j doesn't know what type of player i he is facing. He considers perhaps several types of player i as possibilities for his opponent. He needs to consider the actions of all those types of player i, because he can't rule any of them out. Since every player must be able to compute a best response, every player must know the planned actions of all types of all other players. Therefore any well-defined strategy profile must define an action for every type of every player. Therefore a strategy profile s maps type profiles into action profiles; i.e. $s{:}\,\Theta\to A$, $s\in S=\mathsf{X}_{i\in I}S_i$.

Bayesian equilibrium

Consider a particular player $i\in I$ and a particular one of her types $\theta_i\in\Theta_i$. Assume that her $n-1$ opponents' types are described by some deleted type profile $\theta_{-i}\in\Theta_{-i}$ and that they play some deleted action profile $a_{-i}\in A_{-i}$. If player i then chooses an action $a_i\in A_i$, her utility will be $u_i((a_i,a_{-i}),(\theta_i,\theta_{-i}))$. More generally, if the players choose a mixed-action profile $\alpha\in\mathcal{A}$, player i's expected utility is $u_i((\alpha_i,\alpha_{-i}),(\theta_i,\theta_{-i}))$.

Now assume player i knows the type-contingent mixed strategies $\sigma_{-i}\in\Sigma_{-i}$ her opponents are playing; i.e. she knows what mixed actions they would take for any given set of types. However, she doesn't know their *realized* types, so she doesn't know the actual deleted mixed-action profile α_{-i} which will occur as a result of their type-contingent strategies. What action $a_i\in A_i$ should player i choose? Although player i doesn't know θ_{-i}, she does know the probability distribution p by which Nature generates type profiles; and she also knows her own type θ_i, upon which she conditions her subjective probability about the types θ_{-i} of her opponents. For any particular combination θ_{-i} of other players' types, player i assesses this combination the probability $\hat{p}_i(\theta_{-i}\,|\,\theta_i)$. Therefore she also adds this probability to the event that her opponents will choose the particular deleted mixed-action profile $\sigma_{-i}(\theta_{-i})\in\mathcal{A}_{-i}$. Player i's expected utility, then, given her knowledge of her own type θ_i and of her opponents' type-contingent strategies σ_{-i}, if she chooses the action $a_i\in A_i$, is

$$\sum_{\theta_{-i}\in\Theta_{-i}} \hat{p}_i(\theta_{-i}\,|\,\theta_i)\,u_i((a_i,\sigma_{-i}(\theta_{-i})),(\theta_i,\theta_{-i})). \tag{5}$$

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8 A^B is the set of all functions from $B\to A$.

For a_i to be a best response by type θ_i of player i, that choice must maximize (5) over her action space A_i. We define player i's best-response correspondence $\mathsf{BR}_i: \Sigma_{-i} \times \Theta_i \to A_i$, which maps opponents' strategy profiles and player-i types into player-i actions, by

$$\mathsf{BR}_i(\sigma_{-i}, \theta_i) = \arg\max_{a_i \in A_i} \sum_{\theta_{-i} \in \Theta_{-i}} \hat{p}_i(\theta_{-i} \mid \theta_i) u_i((a_i, \sigma_{-i}(\theta_{-i})), (\theta_i, \theta_{-i})). \tag{6}$$

When a strategy profile $\sigma \in \Sigma$ is such that every type of every player is maximizing her expected utility given the type-contingent strategies of her opponents, then we say that σ is a Bayesian-Nash equilibrium of this game of incomplete information. In other words, σ is a *Bayesian-Nash equilibrium* if $\forall i \in I$, $\forall \theta_i \in \Theta_i$, $\mathsf{supp}\, \sigma_i(\theta_i) \subset \mathsf{BR}_i(\sigma_{-i}, \theta_i)$.

Example: To build or not to build

Consider two firms and one market.[9] Firm 1 will definitely produce in this market; firm 2 may decide to Enter the market or Refrain. Firm 1 faces an investment decision: should it make a costly investment in order to modernize its plant? If firm 1 modernizes, firm 2 will find the competition overwhelming and would be better off not entering. However, if firm 1 eschews investment—keeping antique technology—firm 2 would find it worthwhile to enter and compete with firm 1. Firm 1's investment cost is either low or high; firm 1 knows this cost, but firm 2 only has beliefs such that firm 1's cost is high with probability ρ and low with probability $1-\rho$. The payoffs are shown in Figure 1.

High investment cost $[\rho]$

	Enter $[y]$	Refrain $[1-y]$
Modern	0,−2	4,**0**
Antique	**4**,**2**	**6**,0

Low investment c $[1-\rho]$

	Enter $[y]$	Refrain $[1-y]$
Modern $[x]$	3,−2	**7**,**0**
Antique $[1-x]$	**4**,**2**	6,0

Figure 1: A Bayesian investment/entry game.

Firm 2 has no private information, therefore its type space Θ_2 is degenerate—a singleton—so we will ignore it. Firm 1 has private information about its cost, so its type space has two elements, which we'll denote $\underline{c}$ and $\bar{c}$, for low and high cost, respectively. A strategy for firm 1 is an action for each of its two types, i.e. $s_1(c) \in A_1 = \{M, A\}$, for $c \in \Theta_1 = \{\underline{c}, \bar{c}\}$. A strategy for firm 2 is just a single action $a_2 \in A_2 = \{E, R\}$.

We note immediately that the high-cost firm 1 has a strictly dominant action, viz. A. Therefore in any Bayesian equilibrium s we must have $s_1(\bar{c}) = A$. The best-response for the low-cost firm 1, however, depends on Firm 2's strategy. If firm 2 enters, firm 1 prefers A; if firm 2 refrains, firm 1 prefers M. We denote by y the probability that firm 2 chooses to Enter. The low-cost firm 1's payoffs to Modern and Antique, respectively, as a function of firm 2's mixed strategy y, are

9 See Fudenberg and Tirole [1991: →6.2, 211–213] for an example of Bayesian equilibrium with a continuum of types.

$$u_1(M; y, \underset{\sim}{c}) = 3y + 7(1-y) = 7 - 4y,$$

$$u_1(A; y, \underset{\sim}{c})) = 4y + 6(1-y) = 6 - 2y.$$

The low-cost firm 1 weakly prefers M, then, when $y \leq \frac{1}{2}$. Denoting by x the probability that the low-cost firm 1 chooses M, we can write the low-cost firm 1's mixed-strategy best-response correspondence $x^*(y)$ as

$$x^*(y) = \begin{cases} \{1\}, & y < \frac{1}{2}, \\ [0,1], & y = \frac{1}{2}, \\ \{0\}, & y > \frac{1}{2}. \end{cases}$$

Now we seek Firm 2's best-response correspondence. Firm 2 faces the high-cost firm 1 with probability ρ, in which case Firm 1 definitely chooses A, resulting in a payoff of 2 to firm 2 if he enters. With probability $1-\rho$, Firm 2 faces the low-cost firm 1, who chooses M with some probability x. If Firm 2 chooses to Enter, its expected payoff is

$$u_2(E; x) = 2\rho + (1-\rho)[-2x + 2(1-x)] = 2 - 4(1-\rho)x.$$

If Firm 2 chooses Refrain, its expected payoff is zero. Firm 2 weakly prefers to Enter, then, if $x \leq 1/2(1-\rho) \equiv \bar{x}$. We note that $\bar{x} \in [\frac{1}{2}, \infty)$ for $\rho \in [0,1]$. We can write Firm 2's mixed-strategy best-response correspondence, then, as

$$y^*(x) = \begin{cases} \{1\}, & x < \bar{x}, \\ [0,1], & x = \bar{x}, \\ \{0\}, & x > \bar{x}. \end{cases}$$

We note that, unlike the low-cost firm 1's best-response correspondence x^*, firm 2's best-response correspondence y^* depends on ρ through y^*'s dependence upon $\bar{x}$.

Now we can easily find the Bayesian-Nash equilibria of the game determined by the high-cost probability ρ by finding the intersection of the graphs of the players' mixed-strategy best-response correspondences. Figure 2 plots the graphs of x^* and y^* for three cases: a when $\rho \in [0, \frac{1}{2})$, b when $\rho = \frac{1}{2}$, and c when $\rho \in (\frac{1}{2}, 1]$. The Bayesian-Nash equilibria are indicated with ☆s. We can write any equilibrium of this Bayesian game in the form of a triple:

$$(A, x \circ M \oplus (1-x) \circ A; y \circ E \oplus (1-y) \circ W),$$

where the first element is the high-cost firm 1's dominant action A, the second element is the low-cost firm 1's mixed action, and the third element (separated by a semicolon to indicate a different player) is firm 2's mixed action.

When firm 1 is more likely to be low cost, viz. when $\rho < \frac{1}{2}$, the character of the equilibria is determined by the low-investment-cost game on the right-hand side of Figure 1. There are two pure-strategy equilibria $(A,M;R)$ and $(A,A;E)$, corresponding respectively to 1 modernization by the low-cost firm 1 and refraining from entry by firm 2 and 2 sticking with antique technology by the low-cost firm 1 and entry by firm 2. There is also a mixed-strategy equilibrium in which the low-cost firm 1 modernizes one-half the time and firm 2 enters with probability $\bar{x}$, which increases the more likely firm 1 is high-cost.

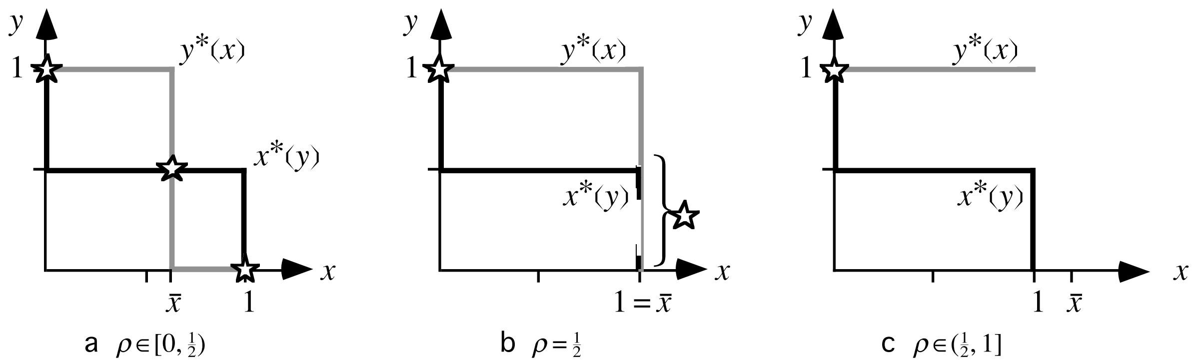

Figure 2: Best-response correspondences for the low-cost firm 1 and firm 2 for three regimes of the probability that firm 1 has high cost.

When it is equally likely that firm 1 is low or high cost, viz. when $\rho=\frac{1}{2}$, there is again a pure-strategy equilibrium in which both types of firm 1 stay with the antique technology and firm 2 enters. The remaining pure-strategy and mixed-strategy equilibria from case a have now merged together into a continuum of equilibria: The low-cost firm 1 modernizes, which makes firm 2 indifferent to entry. As long as firm 2 doesn’t enter with too high a probability, the low-cost firm 1 still strictly prefers to modernize.

When it is more likely that firm 1 is high cost, the Bayesian game is dominated by the high-cost game on the left-hand side of Figure 1. It has a single pure-strategy equilibrium, determined by iterated strict dominance, in which both types of firm 1 stay with the antique technology and firm 2 enters.

References

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